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SUMMARY

The purpose of this report is to derive the electrodynamic boundary conditions, surface tractions, and surface power transfer in complete form for easy application to boundary value problems in magnetofluiddynamics.

Features of this report include:

Derivation of the boundary conditions starts from Maxwell's equations in four-dimensional integral form. The boundary conditions are obtained directly in covariant form and are then translated to three-dimensional language.

The inclusion of surface charge and surface current leads to a general current boundary condition containing a surface curvature term. Surface current is found to have a component normal to the surface due to convection of surface charge in this direction.

A surface form for the electromagnetic momentum and energy conservation laws is derived from the boundary conditions by using the same method by which the volumetric conservation laws are derived from Maxwell's equations. This surface form gives the electromagnetic tractions and energy transfer at the surface in two forms: (a) in terms of the jump in the stress-energy tensor and (b) in terms of surface current, surface charge, and the mean fields across the surface.

A set of identities is obtained for the force density and power-transfer density associated with the antisymmetrical part of the stress-energy tensor. The merit of these identities rests largely on their value as evidence in the controversy over the symmetry of the stress-energy tensor.

INTRODUCTION

Although the electrodynamic equations for moving media were established early in the twentieth century by Minkowski, the corresponding boundary conditions have been in a state of continuing development up to the present time. Maxwell's equations were shown to hold unaltered in moving media, and new velocity-dependent constitutive relations were derived to take account of the motion's influence on the electric and magnetic response of the medium. The constitutive equations - although necessary for the solution of actual problems - do not affect the boundary conditions (which are derived directly from Maxwell's equations) and hence are not relevant in the present derivation.

At a fixed interface the boundary conditions are the familiar set given by King (ref. 1, p. 169) (in the notation used herein)

$$\hat{n} \times \left[\vec{H}\right] = \vec{L} \qquad (\hat{n} \cdot \vec{L} = 0)$$

$$\hat{n} \cdot \left[\vec{D}\right] = \eta$$

$$\hat{n} \times \left[\vec{E}\right] = 0$$

$$\hat{n} \cdot \left[\vec{B}\right] = 0$$

$$\mathrm{div}_{\mathrm{surface}}\vec{L} + \hat{n} \cdot \left[\vec{J}\right] + \frac{\partial \eta}{\partial t} = 0$$

where

$\vec{B}$ magnetic field strength

$\vec{E}$ electric field strength

$\vec{D}$ electric excitation

$\vec{H}$ magnetic excitation

$\vec{J}$ electric current density

η surface charge density

$\vec{L}$ surface current density

$\hat{n}$ local unit normal

$\left[G\right]$ discontinuity $G^+ - G^-$, with $\pm$ signs relative to $\hat{n}$

t time coordinate

But this set of jump conditions does not hold for the large class of problems involving moving boundaries. Included in this class are such problems as electromagnetic fields coupled to surface waves on a fluid, moving shock waves in a plasma, and moving solid bodies.

The early derivations of jump conditions at a moving boundary are described in detail by Pauli (ref. 2, pp. 103-104) and Sommerfeld (ref. 3, pp. 285-288). These produced formulas for certain cases but failed to yield general conditions of the type given for boundaries at rest. The present approach is different from early derivations and follows a procedure first applied to electrodynamics by Luneburg in 1944 (ref. 4, pp. 15-22) and later by Truesdell and Toupin in 1958 (ref. 5, pp. 669 and 676-677). Neither of these works included the effects of surface charge density η and surface current density $\vec{L}$ which are essential in many applications. Extending the derivations to include these surface densities is the prime concern of this paper.

The procedure is inherently four-dimensional, involving integrations in Minkowski space. Consequently, a suitable approach is to go the full route to special relativity, to start from Maxwell's equations in four-dimensional covariant form, and afterwards to translate the results to three-dimensional language.

The four-dimensional statement of the electrodynamic momentum and energy conservation laws bears formal similarity to Maxwell's four-dimensional equation for $\vec{H}$ and $\vec{D}$. Hence the surface form of the conservation laws is formally similar to the $\vec{H}$ and $\vec{D}$ boundary condition and may be written by inspection. The same is true for a group of identities associated with the antisymmetrical part of the stress-energy tensor.

Although the derivations are new, a number of the simpler results presented here were obtained previously in reference 6 (chs. I and II) by using a three-dimensional technique and the Lorentz transformation.

SYMBOLS AND NOTATION

Mathematical Notation

$\hat{}$	unit spatial vectors
$\rightarrow, \leftrightarrow$	spatial vectors and tensors
*	dual vectors and tensors
$[\;]$	jump across a surface of discontinuity
$\langle\;\rangle$	mean value across a surface of discontinuity
∇	spatial gradient operator, $\hat{i}_1 \frac{\partial}{\partial x} + \hat{i}_2 \frac{\partial}{\partial y} + \hat{i}_3 \frac{\partial}{\partial z}$

$\square$ space-time gradient operator, $\hat{i}_1 \frac{\partial}{\partial x} + \hat{i}_2 \frac{\partial}{\partial y} + \hat{i}_3 \frac{\partial}{\partial z} + \hat{i}_4 \frac{1}{ic} \frac{\partial}{\partial t}$

$| \, |$ modulus of a spatial vector or of a four-vector

$i = \sqrt{-1}$

Superscripts:

a antisymmetric part of stress-energy tensor or a derivative thereof

s symmetric part of stress-energy tensor or a derivative thereof

$\pm$ limiting value as a surface of discontinuity is approached from the $\pm$ sides, with $\pm$ designations relative to surface unit normal

Subscripts:

Subscripts obey the summation convention of Cartesian tensor notation.

Latin subscripts run from 1 to 3 (x,y,z) with the exception of t which represents time.

Greek subscripts run from 1 to 4 (x,y,z,ict).

Commas in subscripts denote partial differentiation. For example, $\varphi_{,i} \equiv \frac{\partial \varphi}{\partial x_i}$.

Symbols

a area segment of a two-dimensional surface, meters^2

$\vec{B}$ magnetic field strength, webers/meter^2

C mean surface curvature, meter^{-1}

c speed of light in vacuum, meters/second

$\vec{D}$ electric excitation, coulombs/meter^2

$\vec{E}$ electric field strength, volts/meter

$F_{\alpha\beta}$ electromagnetic field strength four-tensor, volts/meter

$f_{\alpha\beta}$ electromagnetic excitation four-tensor, amperes/meter

$\vec{H}$ magnetic excitation, amperes/meter

$\vec{J}$ electric current density, coulombs/meter^2-second

$\vec{K}$ electromagnetic surface traction, newtons/meter^2

$\vec{k}$ electromagnetic force density, newtons/meter^3

$\vec{L}$ electric surface current density, coulombs/meter-second

$l_{\alpha\beta}$ Lorentz transformation matrix

l closed contour line bounding area a of a two-dimensional surface, meters

$\vec{M} = \frac{1}{2}\left(\vec{E} \times \vec{D} + \vec{H} \times \vec{B}\right)$, a vector derived from $T^a_{\alpha\beta}$

m_α unit four-normal to the three-dimensional hypersurface $\varphi(x,y,z,t) = 0$ in space-time

N normal velocity, or speed of displacement, of a surface of electromagnetic discontinuity, meters/second

n_i (or $\hat{n}$) unit normal to a two-dimensional surface

P surface power transfer from electromagnetic fields to other forms of energy, watts/meter^2

p volumetric power transfer from electromagnetic fields to other forms of energy, watts/meter^3

$\vec{R} = \frac{1}{2}\left(\vec{D} \times \vec{B} - \frac{1}{c^2}\,\vec{E} \times \vec{H}\right)$, a vector derived from $T^a_{\alpha\beta}$

S closed three-dimensional hypersurface in space-time, meters^3

T_{ij} Maxwell stress tensor, $E_iD_j + H_iB_j - W\delta_{ij}$, or spatial part of $T_{\alpha\beta}$, newtons/meter^2

$T_{\alpha\beta}$ electrodynamic stress-energy tensor, newtons/meter^2

t time coordinate, second

V four-dimensional hypervolume in space-time, meters^4

W electrodynamic energy density, $\frac{1}{2}\left(\vec{H} \cdot \vec{B} + \vec{D} \cdot \vec{E}\right)$, joules/meter^3

x_i Cartesian spatial coordinates (x,y,z), meters

x_α Cartesian space-time coordinates (x,y,z,ict), meters

Γ_α electric four-current density $(J_1,J_2,J_3,ic\rho)$, coulombs/meter^2-second

δ_{ij} Kronecker delta

$\delta_{\alpha\beta\gamma\epsilon}$ Levi-Civita symbol defined by equation (31)

ϵ_o electric permittivity of vacuum, farads/meter

ϵ_{ijk} Levi-Civita symbol defined by equation (C6)

η electric surface charge density, coulombs/meter^2

K_α electromagnetic surface quasi-four-force density $\left(K_1,K_2,K_3,\frac{i}{c}P\right)$, newtons/meter^2

κ_α electromagnetic four-force density $\left(k_1,k_2,k_3,\frac{i}{c}p\right)$, newtons/meter^3

Λ_α electric surface quasi-four-current density $\left(L_1,L_2,L_3,ic\eta\right)$, coulombs/meter-second

μ_o magnetic permeability of vacuum, henrys/meter

μ_α outward unit four-normal on a closed three-dimensional hypersurface S in space-time

ρ electric charge density, coulombs/meter^3

σ three-dimensional area on hypersurface $\varphi(x,y,z,t) = \text{Constant}$ in space-time, meters^3

τ small dimensionless parameter

$\varphi(x,y,z,t)$ a continuously twice-differentiable function; $\varphi = \text{Constant}$ represents a moving and deforming surface in space (fig. 1) and a three-dimensional hypersurface in space-time (fig. 2)

Ψ_α^* four-vector dual to tensor $\psi_{\lambda\mu,\nu}$

$\psi_{\alpha\beta}$ arbitrary tensor

I. ELECTRODYNAMIC JUMP CONDITIONS AT A MOVING BOUNDARY

General Considerations

Moving surfaces.- Consider a two-dimensional surface in arbitrary motion defined by $\varphi(x,y,z,t) = 0$, where φ is a continuously twice-differentiable function of Cartesian spatial and time coordinates. (See fig. 1.) The local unit normal to the surface is given by

$$n_i = \frac{\varphi_{,i}}{|\nabla\varphi|} \qquad (1)$$

provided $|\nabla\varphi|$ does not vanish on the boundary, where Latin subscripts run from 1 to 3, and where commas in the subscript denote partial differentiation.

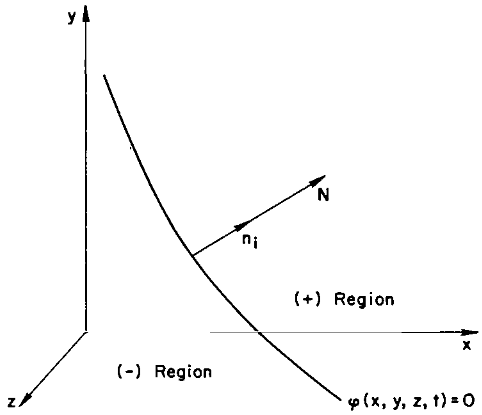

Figure 1. - Two-dimensional surface (edge view) moving through space with local speed of displacement N.

The local normal velocity of the surface, which is called the local speed of displacement, or simply the speed, of the surface may be determined as in Truesdell and Toupin (ref. 5, pp. 498-499): For a point which moves with the surface, the total differential

$$d\varphi = \varphi_{,j}\, dx_j + \varphi_{,t}\, dt = 0$$

Transposing terms and dividing by $|\nabla\varphi|\, dt$ yields

$$n_j \frac{dx_j}{dt} = -\frac{\varphi_{,t}}{|\nabla\varphi|}$$

The coordinates x_j are those of a point which remains on the surface; hence, the left-hand side of this equation is the surface speed, which is denoted N in order to avoid confusion with the usual notation for fluid velocity. Thus, the local speed of a surface $\varphi(x,y,z,t) = 0$ is given by

$$N = -\frac{\varphi_{,t}}{|\nabla\varphi|} \qquad (2)$$

where $N > 0$ for motion in the sense of $\hat{n}$ (eq. (1)). Equations (1) and (2) may be used to write the total differential $d\varphi$ as

$$d\varphi = |\nabla\varphi|\left(n_j\, dx_j - N\, dt\right) \qquad (3)$$

Take the moving surface $\varphi = 0$ to be a surface of electromagnetic discontinuity, across which electric fields $\vec{E}$ and $\vec{D}$, magnetic fields $\vec{B}$ and $\vec{H}$, current density $\vec{J}$, and charge density ρ may have finite increments. Surface current density $\vec{L}$ and surface charge density η may also be prescribed on the surface.

It is perhaps worthwhile to note that the moving surface referred to here is an abstract geometrical entity and quite independent (in concept) of the material medium or of its motion. For example, the medium on one side (or on both sides) may be in tangential motion while the surface remains at rest, as in steady, inviscid flow. Again, the medium on either side may have normal velocity components while the speed of the surface is zero, as with a stationary shock wave. Moreover, discontinuities in the medium are not necessary for the fields to be discontinuous. Surfaces of electromagnetic discontinuity can occur within a homogeneous medium, as at the front of a step-shaped light wave, or at the Čerenkov light cone of an electron which exceeds the speed of light in the medium.

It is clearly the motion of the electromagnetic discontinuity surface which is directly relevant here - not the motion of matter. Hence, without restricting the latter, it seems reasonable that N, the surface speed of displacement, is the only velocity that will appear in the boundary conditions. This viewpoint has previously been expressed by Goldstein (ref. 7, pp. 69-70).

Now view the surface $\varphi = 0$ in Minkowski space - the four-dimensional space-time continuum of special relativity - and take the coordinate axes to be x,y,z,ict. In this representation $\varphi = 0$ is a three-dimensional hypersurface in space-time. (See fig. 2.) The unit four-normal to this hypersurface is given by

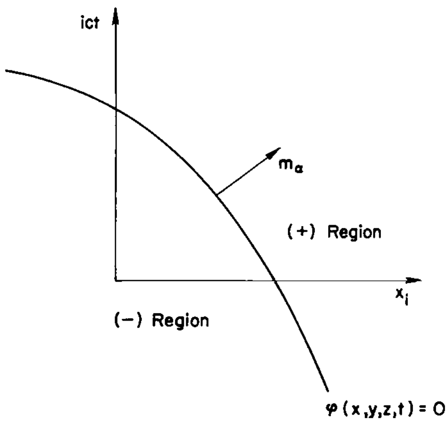

Figure 2.- Three-dimensional hypersurface in space-time (edge view with spatial coordinates contracted).

$$m_\alpha = \frac{\varphi_{,\alpha}}{|\Box\varphi|} \tag{4}$$

where Greek subscripts run from 1 to 4, and $\Box$ is the four-dimensional gradient defined by

$$\Box \equiv \hat{i}_1 \frac{\partial}{\partial x} + \hat{i}_2 \frac{\partial}{\partial y} + \hat{i}_3 \frac{\partial}{\partial z} + \hat{i}_4 \frac{1}{ic} \frac{\partial}{\partial t} \tag{5a}$$

The denominator in equation (4) is real and positive, since by equations (1) and (2)

$$|\Box\varphi| = \sqrt{\varphi_{,\beta}\varphi_{,\beta}} = |\nabla\varphi|\sqrt{1 - \frac{N^2}{c^2}} \tag{5b}$$

In four-vector notation the total differential $d\varphi$ becomes

$$d\varphi = \varphi_{,\beta}\, dx_\beta = |\Box\varphi| m_\beta\, dx_\beta \tag{6}$$

Maxwell's equations.- Maxwell's equations in four-dimensional integral form (with Cartesian tensor notation) are

$$\oiiint f_{\alpha\beta}\mu_\beta\, dS = \iiiint \Gamma_\alpha\, dV \tag{7a}$$

or

$$\left.\begin{aligned} &\oiiint \left(F_{\alpha\beta}\mu_\gamma + F_{\beta\gamma}\mu_\alpha + F_{\gamma\alpha}\mu_\beta\right) dS = 0 \\ &\oiiint F^*_{\alpha\beta}\mu_\beta\, dS = 0 \end{aligned}\right\} \tag{7b}$$

$$\oiiint \Gamma_\beta\mu_\beta\, dS = 0 \tag{7c}$$

where V is an arbitrary four-dimensional hypervolume in space-time; S is the closed three-dimensional hypersurface bounding V; μ_β is the outward unit four-normal on S; and

$$x_\alpha = (x,y,z,ict) \tag{8a}$$

$$\Gamma_\alpha = \left(J_1,J_2,J_3,ic\rho\right) \tag{8b}$$

$$f_{\alpha\beta} = \begin{pmatrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & -H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{pmatrix} \tag{8c}$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & cB_3 & -cB_2 & -iE_1 \\ -cB_3 & 0 & cB_1 & -iE_2 \\ cB_2 & -cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \tag{8d}$$

$$F^*_{\alpha\beta} = \begin{pmatrix} 0 & -iE_3 & iE_2 & cB_1 \\ iE_3 & 0 & -iE_1 & cB_2 \\ -iE_2 & iE_1 & 0 & cB_3 \\ -cB_1 & -cB_2 & -cB_3 & 0 \end{pmatrix} \tag{8e}$$

The first of equations (7b) has but four independent components obtained by setting $\alpha\beta\gamma$ = 123, 124, 134, and 234. Equation (7c) is a direct consequence of equation (7a) and the antisymmetry of $f_{\alpha\beta}$; $F^*_{\alpha\beta}$ is the dual tensor of $F_{\alpha\beta}$. Formally the metric is a definite form, since in terms of the coordinates (8a) $ds^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + dx_4{}^2$, where ds is the length of a line element in space-time.

Maxwell's equations, together with the relation $F_{\alpha\beta} = \sqrt{\frac{\mu_0}{\epsilon_0}}\, f_{\alpha\beta}$ (in vacuum), constitute the axioms of electrodynamics. As written here (in the notation of special relativity) they apply to inertial reference frames and, in particular, to the laboratory frame. It is assumed, though, that the curvature of space-time is negligible. Aside from this assumption, no constraints are placed on the material medium, its state of motion, anisotropy, temperature, mechanical stress, or the like. (Properties of the medium appear in the constitutive relations, which the ensuing treatment is not concerned with.)

The axioms have, of course, been recast into a number of varied forms. (See Truesdell and Toupin (ref. 5, pp. 666-668), Sommerfeld (ref. 3, pp. 286-287), and Fano, Chu, and Adler (ref. 8, pp. 389, 480, and 483).) The procedure which follows can be applied to each of these.

Hilbert's view that all natural laws should be expressed in integral form is adhered to in the preceding formulation. (See ref. 5, p. 232.) In integral form the axioms have uniform applicability. In regions of regularity they are identical with Maxwell's equations in differential form, which are obtained from set (7) by using the divergence theorem

$$f_{\alpha\beta,\beta} = \Gamma_\alpha \tag{9a}$$

or

$$\left.\begin{aligned} F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} &= 0 \\ F^*_{\alpha\beta,\beta} &= 0 \end{aligned}\right\} \tag{9b}$$

$$\Gamma_{\beta,\beta} = 0 \tag{9c}$$

(Set (9) is presented by Sommerfeld (ref. 3, pp. 216-218).) At surfaces of discontinuity the axioms give the jump conditions or boundary conditions which shall now be derived.

Derivation of the Boundary Conditions

for Electric and Magnetic Fields

The four-dimensional procedure to be followed was first applied to the boundary conditions for electric and magnetic fields by Luneburg (ref. 4) and later by Truesdell and Toupin (ref. 5). The following treatment is more general since it includes surface charge and surface current which are essential in many applications.

Surface current and charge.- Surface charge density η appears, for example, whenever conductors are placed in an electric field. Surface current density $\vec{L}$ arises by convection of η or by conduction in the limit of infinite conductivity or zero skin depth. For stationary surfaces $\vec{L}$ is locally tangential; but for moving surfaces it seems likely that $\vec{L}$ should have a normal component arising from convection of η in the normal direction.

A surface quasi-four-current density

$$\Lambda_\alpha = \left(L_1, L_2, L_3, ic\eta\right) \tag{10a}$$

may be defined for a fixed surface $\varphi(x,y,z) = 0$ by

$$\Lambda_\alpha = \lim_{\tau\to 0} \int_{x_j(-\tau)}^{x_j(\tau)} \Gamma_\alpha n_j \, dx_j$$

where n_j is the local unit normal and the end points of the line integral lie on the surfaces $\varphi(x_j) = \pm\tau$. But for a moving surface the integration must be made over a moving line element which follows the surface through displacements in the normal direction, and the definition for Λ_α, with reference to figure 3, then becomes

$$\Lambda_\alpha = \lim_{\tau\to 0} \int_{x_j,t(-\tau)}^{x_j,t(\tau)} \Gamma_\alpha \left(n_j \, dx_j - N \, dt\right) \tag{10b}$$

where the end points of the line integral lie on the moving surfaces $\varphi\left(x_j,t\right) = \pm\tau$. Alternately, from equation (3)

$$\Lambda_\alpha = \lim_{\tau\to 0} \int_{-\tau}^{\tau} \Gamma_\alpha \frac{d\varphi}{|\nabla\varphi|} \tag{10c}$$

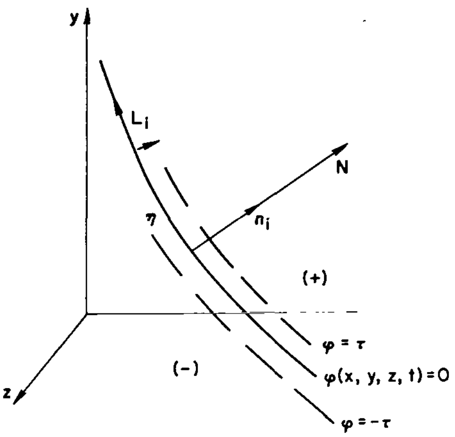

Figure 3. - Two-dimensional moving surface in space (edge view) with surface-charge density η and surface-current density L_i shown.

The term Λ_α is called a quasi-four-vector because the product $|\nabla\varphi|\Lambda_\alpha$ is a Lorentz invariant, although Λ_α itself is not. These transformation properties are presented in appendix A.

<u>Boundary conditions on $\vec{H}$ and $\vec{D}$.</u>- In axiom (7a) choose V to be a segment of the hypervolume included between hypersurfaces $\varphi = \pm\tau$, as illustrated in figure 4, and take the limit as $\tau \to 0$

$$\lim_{\tau\to 0} \oiiint f_{\alpha\beta}\mu_\beta \, dS = \lim_{\tau\to 0} \iiiint \Gamma_\alpha \, dV \tag{11}$$

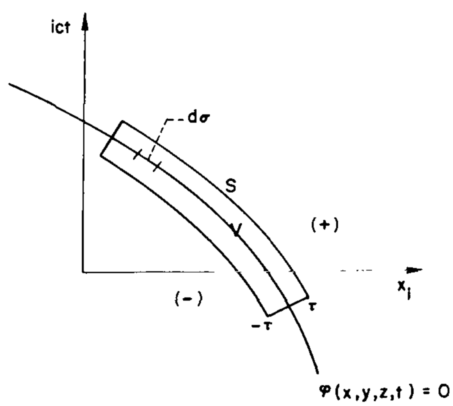

Figure 4.- Four-dimensional hypervolume V enclosed by three-dimensional hypersurface S in space-time (viewed along constant φ hypersurfaces with spatial dimensions contracted).

If $d\sigma$ denotes an element of the hypersurface $\varphi = 0$ and $m_\beta \, dx_\beta$ denotes a line element along its unit four-normal m_β, the volume integral may be written in the limit as an integration over σ and its four-normal

$$\lim_{\tau\to 0} \iiiint \Gamma_\alpha \, dV = \lim_{\tau\to 0} \iiint d\sigma \int_{x_\beta(-\tau)}^{x_\beta(\tau)} \Gamma_\alpha m_\beta \, dx_\beta \tag{12}$$

The integral over S may also be changed to an integral over σ, since the edge contributions of S vanish in the limit, and equation (11) becomes

$$\iiint \left[f_{\alpha\beta}\right] m_\beta \, d\sigma = \lim_{\tau\to 0} \iiint d\sigma \int_{x_\beta(-\tau)}^{x_\beta(\tau)} \Gamma_\alpha m_\beta \, dx_\beta \tag{13}$$

where m_β is the unit four-normal to $\varphi = 0$ given by equation (4), and $[G]$ denotes the discontinuity $G^+ - G^-$ at $\varphi = 0$ with ± signs relative to m_β.

Using equation (6) to change variables in the line integral on the right of equation (13) gives

$$\iiint [f_{\alpha\beta}] m_\beta \, d\sigma = \lim_{\tau \to 0} \iiint d\sigma \int_{-\tau}^{\tau} \Gamma_\alpha \frac{d\varphi}{|\Box\varphi|} \tag{14}$$

and by definition (10c)

$$\iiint [f_{\alpha\beta}] m_\beta \, d\sigma = \iiint d\sigma \, \Lambda_\alpha \frac{|\nabla\varphi|}{|\Box\varphi|} \tag{15}$$

where Λ_α is the surface quasi-four-current density (eq. (10a)). Since the remaining integration is over an arbitrary portion of the hypersurface $\varphi = 0$ and the integrands are assumed to be continuous along the hypersurface, the integral signs may be dropped and the equation written

$$[f_{\alpha\beta}] m_\beta = \Lambda_\alpha \frac{|\nabla\varphi|}{|\Box\varphi|} \tag{16}$$

on $\varphi = 0$. Substitution of equation (4) gives

$$[f_{\alpha\beta}] \varphi_{,\beta} = \Lambda_\alpha |\nabla\varphi| \tag{17}$$

on $\varphi = 0$, and this is the boundary condition derived from axiom (7a) in covariant four-dimensional form.

This boundary condition has a corollary which is readily obtained by multiplying both sides of equation (17) by $\varphi_{,\alpha}$. The left-hand side of this equation vanishes since $f_{\alpha\beta}$ is an antisymmetric tensor, and the corollary is obtained in covariant form

$$\varphi_{,\alpha} \Lambda_\alpha = 0 \tag{18}$$

on $\varphi = 0$.

Condition (17) and its corollary (18) may be translated to three-dimensional form by expanding the left-hand sides of both equations and dividing by $|\nabla\varphi|$:

$$\left[f_{\alpha j}\right]\frac{\varphi_{,j}}{|\nabla\varphi|} + \left[f_{\alpha 4}\right]\frac{\varphi_{,t}}{ic|\nabla\varphi|} = \Lambda_\alpha \tag{19}$$

$$\Lambda_j \frac{\varphi_{,j}}{|\nabla\varphi|} + \Lambda_4 \frac{\varphi_{,t}}{ic|\nabla\varphi|} = 0 \tag{20}$$

By definitions (1) and (2) these equations become

$$\left[f_{\alpha j}\right]n_j - \left[f_{\alpha 4}\right]\frac{N}{ic} = \Lambda_\alpha \tag{21}$$

$$\Lambda_j n_j - \Lambda_4 \frac{N}{ic} = 0 \tag{22}$$

The function φ no longer appears explicitly, and $[G]$ denotes the discontinuity $G^+ - G^-$ with $\pm$ signs relative to the spatial normal n_j. Substitution of the field matrices (8c) and (10a) gives for the first three components of condition (21)

$$\hat{n} \times \left[\vec{H}\right] + N\left[\vec{D}\right] = \vec{L} \tag{23a}$$

for its fourth component

$$\hat{n} \cdot \left[\vec{D}\right] = \eta \tag{23b}$$

and for the corollary (eq. (22))

$$\hat{n} \cdot \vec{L} - N\eta = 0 \tag{24}$$

at a moving surface of discontinuity.

<u>Boundary conditions on $\vec{E}$ and $\vec{B}$</u>.- Axiom (7b) is clearly amenable to the foregoing treatment, and the resultant boundary condition in either of two covariant forms may now be written by inspection as

$$\left.\begin{aligned} &\left[F_{\alpha\beta}\right]\varphi_{,\gamma} + \left[F_{\beta\gamma}\right]\varphi_{,\alpha} + \left[F_{\gamma\alpha}\right]\varphi_{,\beta} = 0 \\ \text{or} \quad &\left[F^*_{\alpha\beta}\right]\varphi_{,\beta} = 0 \end{aligned}\right\} \tag{25}$$

on $\varphi = 0$. Translation of condition (25) to three dimensions gives

$$\hat{n} \times \left[\vec{E}\right] - N\left[\vec{B}\right] = 0 \tag{26a}$$

$$\hat{n} \cdot \left[\vec{B}\right] = 0 \tag{26b}$$

at a moving surface of discontinuity.

Discussion.- A comparison of boundary condition (23a) with the corresponding Maxwell equation $\operatorname{curl} \vec{H} - \vec{D}_{,t} = \vec{J}$ shows at once that the term $-N\left[\vec{D}\right]$ represents a surface displacement current density arising from the rapid change in $\vec{D}$ which a fixed observer would note as the moving surface passes him by. Similarly, the term $-N\left[\vec{B}\right]$ of boundary condition (26a) is the surface counterpart of $\vec{B}_{,t}$ from Maxwell's equation $\operatorname{curl} \vec{E} + \vec{B}_{,t} = 0$.

The result that N, the speed of displacement of the electromagnetic-discontinuity surface, is the only velocity that appears in the boundary conditions (although the material medium is in no way restricted to purely normal motions) was anticipated in the section entitled "Moving surfaces." Also expected was equation (24) for the normal component of surface-current density $\vec{L}$ arising from convection of surface-charge density η in the normal direction.

Derivation of the Current Boundary Condition

Four-dimensional form.- The current boundary condition at a surface of discontinuity which carries surface charge density η and surface current density $\vec{L}$ evolves from axiom (7c)

$$\oint\!\!\!\oint\!\!\!\oint \Gamma_\alpha \mu_\alpha \, dS = 0$$

with the closed hypersurface S as depicted in figure 4. It is convenient to split the integral as shown:

$$\iiint_{S_{\parallel}} \Gamma_\alpha \mu_\alpha \, dS + \iiint_{S_{\perp}} \Gamma_\alpha \mu_\alpha \, dS = 0 \tag{27}$$

where $S_{\parallel}$ denotes the two parts of S parallel to the hypersurface $\varphi = 0$ and $S_{\perp}$ is the hypercylindrical part of S which cuts $\varphi = 0$ and is orthogonal to it.

The intersection of hypercylinder $S_\perp$ with hypersurface $\varphi = 0$ is a closed two-dimensional hypercircuit which surrounds a region σ on $\varphi = 0$. In the limit as $\tau \to 0$ the integral over $S_{\parallel}$ may be written as an integral over σ, and equation (27) becomes

$$\iiint \left[\Gamma_\alpha\right] m_\alpha \, d\sigma + \lim_{\tau \to 0} \iiint_{S_\perp} \Gamma_\alpha \mu_\alpha \, dS = 0 \tag{28}$$

where m_α is the unit four-normal to hypersurface $\varphi = 0$ defined by equation (4).

The integral over $S_\perp$ in equation (28) does not vanish here as $\tau \to 0$, in contrast to the previous derivation (eqs. (11) to (13)) since the defining equation (10c) for $\Lambda_\alpha = \left(L_1, L_2, L_3, ic\eta\right)$ implies that Γ_α is infinite on $\varphi = 0$. Evaluation of the integral is begun by defining an orthogonal set of four line elements

$$\delta x_\mu \qquad \Delta x_\nu \qquad m_\alpha m_\epsilon \, dx_\epsilon \qquad \mu_\alpha \mu_\epsilon \, dx_\epsilon \tag{29}$$

where

$\delta x_\mu, \Delta x_\nu$ are the components of two orthogonal line elements which lie along the two-dimensional intersection of hypercylinder $S_\perp$ with hypersurface $\varphi = 0$

$m_\alpha m_\epsilon \, dx_\epsilon$ is a line element along m_α, the unit four-normal on $\varphi = 0$

$\mu_\alpha \mu_\epsilon \, dx_\epsilon$ is a line element along μ_α, the outward unit four-normal on $S_\perp$

The relevant hypersurface elements written in terms of set (29) are

$$m_\theta \, d\sigma = \delta_{\theta\lambda\mu\nu}\left(\mu_\lambda \mu_\epsilon \, dx_\epsilon\right)\delta x_\mu \Delta x_\nu \qquad (\text{on } \varphi = 0) \tag{30a}$$

$$\mu_\alpha \, dS = -\delta_{\alpha\beta\mu\nu}\left(m_\beta m_\gamma \, dx_\gamma\right)\delta x_\mu \Delta x_\nu \qquad (\text{on } S_\perp) \tag{30b}$$

where the Levi-Civita symbol

$$\delta_{\alpha\beta\gamma\epsilon} = \begin{cases} +1(-1) & (\text{for } \alpha\beta\gamma\epsilon \text{ an even (odd) permutation of } 1234) \\ 0 & (\text{for two or more indices alike}) \end{cases} \tag{31}$$

The line elements δx_μ and Δx_ν have circumferential directions on the hypercylinder $S_\perp$ and may be combined in a tensor element $ds_{\mu\nu}$ given by the determinant

$$ds_{\mu\nu} = \begin{vmatrix} \delta x_\mu & \Delta x_\mu \\ \delta x_\nu & \Delta x_\nu \end{vmatrix}$$

so that

$$\mu_\alpha \, dS = -\frac{1}{2} \delta_{\alpha\beta\mu\nu}\left(m_\beta m_\gamma \, dx_\gamma\right) ds_{\mu\nu} \qquad (\text{on } S_\perp) \qquad (32)$$

Thus equation (28) becomes

$$\iiint \left[\Gamma_\alpha\right] m_\alpha \, d\sigma - \lim_{\tau \to 0} \frac{1}{2} \oint\!\!\oint ds_{\mu\nu} \int_{x_\gamma(-\tau)}^{x_\gamma(\tau)} \delta_{\mu\nu\alpha\beta} \Gamma_\alpha m_\beta m_\gamma \, dx_\gamma = 0 \qquad (33)$$

and from equations (6) and (10c)

$$\iiint \left[\Gamma_\alpha\right] m_\alpha \, d\sigma - \frac{1}{2} \oint\!\!\oint ds_{\mu\nu} \delta_{\mu\nu\alpha\beta} \Lambda_\alpha m_\beta \frac{|\nabla\varphi|}{|\Box\varphi|} = 0 \qquad (34)$$

Here the second integral is over the closed hypercircuit on $\varphi = 0$ which bounds region σ of the first integral. The second integral can be transformed to an integral over σ by Stokes' theorem, one form of which states (app. B)

$$\oint\!\!\oint ds_{\mu\nu} \psi_{\mu\nu} = \iiint d\sigma \, m_\theta \delta_{\theta\lambda\mu\nu} \psi_{\mu\nu,\lambda} \qquad (35)$$

This transformation is carried out in appendix B, and the result obtained is

$$\iiint \left[\Gamma_\alpha\right] m_\alpha \, d\sigma + \iiint \frac{\left(\Lambda_\alpha |\nabla\varphi|\right)_{,\alpha}}{|\Box\varphi|} \, d\sigma = 0 \qquad (36)$$

on $\varphi = 0$. By the same argument that precedes equation (16),

$$\left[\Gamma_\alpha\right] m_\alpha + \frac{\left(\Lambda_\alpha |\nabla\varphi|\right)_{,\alpha}}{|\Box\varphi|} = 0 \qquad (37)$$

on $\varphi = 0$. Substitution of m_α from equation (4) gives the current boundary condition in covariant four-dimensional form

$$\left[\Gamma_\alpha\right]\varphi_{,\alpha} + \left(\Lambda_\alpha|\nabla\varphi|\right)_{,\alpha} = 0 \tag{38}$$

on $\varphi = 0$. Incidentally, condition (38) can also be obtained by differentiating condition (17), substituting equation (9a), and utilizing the antisymmetry of $f_{\alpha\beta}$.

The remaining task is the translation of this boundary condition to three-dimensional language. For the second term, in particular, this translation requires a close scrutiny of the formal treatment of surface densities.

Functional dependence of surface entities.- It is appropriate, physically, to conceive of the surface densities $\vec{L}$ and η as being defined only on the surface of discontinuity $\varphi = 0$. But this concept is awkward mathematically when the divergence of $\vec{L}$ is required, as in the second term of equation (38). On the other hand, equation (10c), which defines $\Lambda_\alpha = (\vec{L}, ic\eta)$ at the surface $\varphi = 0$, may as readily be applied at any surface of the manifold $\varphi(x,y,z,t) = \text{Constant}$. By this process Λ_α becomes $\Lambda_\alpha(x,y,z,t)$ - a vector field defined throughout a region of space-time. This is the concept adopted here.

To be viewed in like manner are the spatial normal $\hat{n}(x,y,z,t)$ and the surface speed $N(x,y,z,t)$, both of which are derived from the surface function $\varphi(x,y,z,t)$. Ultimately all quantities shall be evaluated at the surface $\varphi = 0$.

Translation to three-dimensional form.- When divided by $|\nabla\varphi|$ and expanded (by using eqs. (1), (2), and (8b)), the first term of condition (38) becomes

$$\left[\Gamma_\alpha\right]\frac{\varphi_{,\alpha}}{|\nabla\varphi|} = \hat{n} \cdot \left[\vec{J}\right] - N\left[\rho\right] \tag{39a}$$

where $[G]$ signifies the jump $G^+ - G^-$ at the surface, with $\pm$ signs relative to the spatial normal $\hat{n}$. Translation of the second term of equation (38) to three-dimensional form (as carried out in appendix C) gives

$$\frac{1}{|\nabla\varphi|}\left(\Lambda_\alpha|\nabla\varphi|\right)_{,\alpha} = \left(\frac{\partial}{\partial t} + N\frac{\partial}{\partial n}\right)\eta + N\eta \operatorname{div}\hat{n} + \hat{n}\cdot\operatorname{curl}(\hat{n}\times\vec{L}) \tag{39b}$$

where $\frac{\partial}{\partial n} = \hat{n} \cdot \nabla$ is the normal derivative and $\left(\frac{\partial}{\partial t} + N\frac{\partial}{\partial n}\right)$ is the displacement derivative following the boundary through displacements along its local normal.

As shown by McConnell (ref. 9, p. 206), the expression $\operatorname{div} \hat{n}$ is related to the mean curvature C of a constant φ surface by

$$C = -\operatorname{div} \hat{n} \tag{40a}$$

where $C > 0$ for concavity in the sense of $\hat{n}$. (Here C is used in place of $2H$, and in Cartesian coordinates, $g^{rs} = \delta_{rs}$.)

The last term of equation (39b) has an alternate form

$$\hat{n} \cdot \operatorname{curl}\left(\hat{n} \times \vec{L}\right) = -\operatorname{div}_{\text{surface}}\left(\hat{n} \times \left(\hat{n} \times \vec{L}\right)\right) \tag{40b}$$

as the two-dimensional surface divergence of the local projection of $\vec{L}$ on a constant φ surface. This identity follows from the theorems of Stokes and Gauss applied to this surface over an arbitrary area a and its bounding contour line l

$$\iint \hat{n} \cdot \operatorname{curl}\left(\hat{n} \times \vec{L}\right) da = \oint \hat{l} \cdot \left(\hat{n} \times \vec{L}\right) dl$$

$$= \oint \left(\hat{l} \times \hat{n}\right) \cdot \vec{L}\, dl$$

$$= -\iint \operatorname{div}_{\text{surface}}\left(\hat{n} \times \left(\hat{n} \times \vec{L}\right)\right) da$$

The local projection of $\vec{L}$ may also be written

$$-\hat{n} \times \left(\hat{n} \times \vec{L}\right) = \vec{L} - \hat{n}N\eta \tag{40c}$$

by corollary (24).

Substitution of expressions (39a) and (39b) into condition (38) with identities (40) completes the translation, and in final three-dimensional form the current boundary condition is

$$\hat{n} \cdot \left[\vec{J}\right] - N\left[\rho\right] + \hat{n} \cdot \operatorname{curl}\left(\hat{n} \times \vec{L}\right) - CN\eta + \left(\frac{\partial}{\partial t} + N \frac{\partial}{\partial n}\right)\eta = 0 \tag{41a}$$

or

$$\hat{n} \cdot \left[\vec{J}\right] - N\left[\rho\right] - \mathrm{div}_{\mathrm{surface}}\left(\hat{n} \times \left(\hat{n} \times \vec{L}\right)\right) - CN\eta + \left(\frac{\partial}{\partial t} + N \frac{\partial}{\partial n}\right)\eta = 0 \qquad (41b)$$

at a moving surface of discontinuity.

The curvature term takes account of changes in the surface area as it moves. Thus for an expanding sphere in vacuum, with a fixed total charge uniformly distributed on its surface, the current condition reduces to $-CN\eta + \left(\frac{\partial}{\partial t} + N \frac{\partial}{\partial n}\right)\eta = 0$. The displacement derivative of η is negative since the charge is distributed over an increasing area; and this is just compensated by the curvature term, where for $\hat{n}$ radially outward N is positive and C, negative.

Sign convention.- A review of the sign convention is given below:

$$\left.\begin{array}{ll}
\text{Four-dimensional form (fig. 2):} & \\
\quad \text{Hypersurface equation} & \varphi(x,y,z,t) = 0 \\
\quad \text{Field discontinuity} & \left[G\right] = G^{+} - G^{-} \text{ with } \pm \text{ signs relative to } \varphi_{,\alpha} \\
\text{Three-dimensional form (fig. 1):} & \\
\quad \text{Surface unit normal} & \hat{n} \\
\quad \text{Surface speed} & N > 0 \text{ for motion in the sense of } \hat{n} \\
\quad \text{Surface curvature} & C > 0 \text{ for concavity in the sense of } \hat{n} \\
\quad \text{Field discontinuity} & \left[G\right] = G^{+} - G^{-} \text{ with } \pm \text{ signs relative to } \hat{n}
\end{array}\right\} \qquad (42)$$

The results of part I are displayed in tables I and II, which show the correlation between the differential and surface forms of Maxwell's equations in four and three dimensions.

TABLE I.- EQUATIONS AND CORRESPONDING BOUNDARY CONDITIONS

IN FOUR-DIMENSIONAL COVARIANT FORM

Maxwell's equations	Jump conditions at hypersurface $\varphi(x,y,z,t) = 0$
$f_{\alpha\beta,\beta} = \Gamma_\alpha$	$\left[f_{\alpha\beta}\right]\varphi_{,\beta} = \Lambda_\alpha\lvert\nabla\varphi\rvert;\ \varphi_{,\beta}\Lambda_\beta = 0$ (Corollary)
$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$	$\left[F_{\alpha\beta}\right]\varphi_{,\gamma} + \left[F_{\beta\gamma}\right]\varphi_{,\alpha} + \left[F_{\gamma\alpha}\right]\varphi_{,\beta} = 0$
or $F^*_{\alpha\beta,\beta} = 0$	or $\left[F^*_{\alpha\beta}\right]\varphi_{,\beta} = 0$
$\Gamma_{\beta,\beta} = 0$	$\left[\Gamma_\beta\right]\varphi_{,\beta} + \left(\Lambda_\beta\lvert\nabla\varphi\rvert\right)_{,\beta} = 0$

TABLE II.- EQUATIONS AND CORRESPONDING BOUNDARY CONDITIONS

IN THREE-DIMENSIONAL FORM

Maxwell's equations	Jump conditions at surface moving with speed N
$\operatorname{curl}\vec{H} - \vec{D}_{,t} = \vec{J}$	$\left.\begin{array}{l}\hat{n}\times\left[\vec{H}\right] + N\left[\vec{D}\right] = \vec{L}\\ \hat{n}\cdot\left[\vec{D}\right] = \eta\end{array}\right\rbrace \hat{n}\cdot\vec{L} - N\eta = 0$ (Corollary)
$\operatorname{div}\vec{D} = \rho$	
$\operatorname{curl}\vec{E} + \vec{B}_{,t} = 0$	$\hat{n}\times\left[\vec{E}\right] - N\left[\vec{B}\right] = 0$
$\operatorname{div}\vec{B} = 0$	$\hat{n}\cdot\left[\vec{B}\right] = 0$
$\operatorname{div}\vec{J} + \rho_{,t} = 0$	$\hat{n}\cdot\left[\vec{J}\right] - N\left[\rho\right] - \operatorname{div}_{\text{surface}}\left(\hat{n}\times\left(\hat{n}\times\vec{L}\right)\right) - CN\eta + \left(\frac{\partial}{\partial t} + N\frac{\partial}{\partial n}\right)\eta = 0$

II. ELECTRODYNAMIC TRACTIONS AND POWER TRANSFER AT A MOVING BOUNDARY

General Considerations

The following identity may be derived directly from Maxwell's equations (9a) and (9b), as in Møller (ref. 10, p. 202):

$$\frac{1}{c} F_{\alpha\theta}\Gamma_\theta + \frac{1}{4c}\left(F_{\theta\gamma}f_{\theta\gamma,\alpha} - F_{\theta\gamma,\alpha}f_{\theta\gamma}\right) = \frac{\partial}{\partial x_\beta}\left(-\frac{1}{c} F_{\alpha\theta}f_{\beta\theta} + \frac{\delta_{\alpha\beta}}{4c} F_{\theta\gamma}f_{\theta\gamma}\right) \qquad (43)$$

where Γ_α, $F_{\alpha\beta}$, and $f_{\alpha\beta}$ are the electromagnetic field tensors defined in set (8). This identity, due originally to Minkowski, may be regarded as a four-dimensional statement of the electrodynamic momentum and energy conservation laws - in which case the differentiated quantity on the right is designated the stress-energy tensor $T_{\alpha\beta}$ and either side represents the electromagnetic four-force density κ_α. Interpreted in this way equation (43) may be written

$$\kappa_\alpha = \frac{1}{c} F_{\alpha\theta}\Gamma_\theta + \frac{1}{4c}\left(F_{\theta\gamma}f_{\theta\gamma,\alpha} - F_{\theta\gamma,\alpha}f_{\theta\gamma}\right) = T_{\alpha\beta,\beta} \qquad (44)$$

where

$$T_{\alpha\beta} = -\frac{1}{c} F_{\alpha\theta}f_{\beta\theta} + \frac{\delta_{\alpha\beta}}{4c} F_{\theta\gamma}f_{\theta\gamma} \qquad (45a)$$

or in matrix form

$$T_{\alpha\beta} = \left(\begin{array}{c|c} E_iD_j + H_iB_j - W\delta_{ij} & -ic\left(\vec{D}\times\vec{B}\right)_i \\ \hline -\frac{i}{c}\left(\vec{E}\times\vec{H}\right)_j & W \end{array}\right) \qquad (45b)$$

with

$$\left.\begin{array}{ll} T_{ij} & \text{Maxwell stress tensor } E_iD_j + H_iB_j - W\delta_{ij} \\ W & \text{electromagnetic energy density } \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right) \\ \vec{D}\times\vec{B} & \text{electromagnetic momentum density} \\ \vec{E}\times\vec{H} & \text{Poynting vector, or electromagnetic energy flux vector} \end{array}\right\} \qquad (45c)$$

The first three components of the four-force density

$$\kappa_\alpha = \left(k_1, k_2, k_3, \frac{i}{c}p\right) \tag{46}$$

give the electromagnetic pondermotive force per unit volume; in the fourth component p is the power per unit volume lost by the fields through conversion to other types of energy (kinetic, thermal, etc.); p is alternately termed the power transfer density.

Incidentally, terms may be added to both sides of equation (43) which preserve the identity but change κ_α and $T_{\alpha\beta}$. (See Møller (ref. 10, pp. 204-206); Pauli (ref. 2, pp. 108-111); and Fano, Chu, and Adler (ref. 8, pp. 492-499).) No general agreement yet exists on which form is correct, but the procedure which follows (treating Minkowski's form) is applicable to each.

Derivation of the Electrodynamic Conservation Laws for a Moving Boundary

A surface form of the momentum and energy conservation laws may be derived from equation (44) by the same procedure used to obtain the boundary conditions from Maxwell's equations in part I. Let $\varphi(x,y,z,t) = 0$ define the moving and deforming surface under consideration. (See fig. 3.) By direct analogy with the surface quasi-four-current density $\Lambda_\alpha = \left(L_1, L_2, L_3, ic\eta\right)$ take

$$K_\alpha = \left(K_1, K_2, K_3, \frac{i}{c}P\right) \tag{47a}$$

to be the surface quasi-four-force density defined by (eq. (10c))

$$K_\alpha = \lim_{\tau \to 0} \int_{-\tau}^{\tau} \kappa_\alpha \frac{d\varphi}{|\nabla\varphi|} \tag{47b}$$

where K_i is the electromagnetic surface traction and P the power transfer per unit area at the surface.

It is clear that equation (44), with its center expression omitted for the moment, is formally similar to Maxwell's equation (9a), and its surface form may be written by inspection (eq. (17))

$$K_\alpha |\nabla\varphi| = \left[T_{\alpha\beta}\right] \varphi_{,\beta} \tag{48}$$

The surface form of the center expression in equation (44) may now be obtained by recalling that the center expression and right-hand side of this identity are related by Maxwell's equations (9a) and (9b); hence, the corresponding terms in the surface form of the identity are related by boundary conditions (17) and (25) - which are the surface forms of (9a) and (9b). Formal substitution of these boundary conditions into the right-hand term of equation (48), as carried out in appendix D, yields the desired center expression

$$K_\alpha = \frac{1}{c}\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta + \frac{1}{4c}\frac{\varphi_{,\alpha}}{|\nabla\varphi|}\left(\left\langle F_{\theta\beta}\right\rangle \left[f_{\theta\beta}\right] - \left[F_{\theta\beta}\right]\left\langle f_{\theta\beta}\right\rangle\right) = \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[T_{\alpha\beta}\right] \qquad (49a)$$

or by identity (D2)

$$K_\alpha = \frac{1}{c}\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta + \frac{1}{4c}\frac{\varphi_{,\alpha}}{|\nabla\varphi|}\left(F^-_{\theta\beta}f^+_{\theta\beta} - F^+_{\theta\beta}f^-_{\theta\beta}\right) = \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[T_{\alpha\beta}\right] \qquad (49b)$$

at $\varphi(x,y,z,t) = 0$, where

$$\langle G\rangle = \frac{1}{2}(G^+ + G^-); \qquad [G] = G^+ - G^- \qquad (49c)$$

with the ± notation defined in set (42). This is the surface conservation law in covariant form. Its translation to three-dimensional language involves the substitution of equations (1), (2), (8c), (8d), (10a), (45b), (45c), and (47a) with the identity $\frac{1}{4c} f_{\theta\beta}F_{\theta\beta} = \frac{1}{2}\left(\vec{H}\cdot\vec{B} - \vec{D}\cdot\vec{E}\right)$. The first three components of equation (49) become

$$\vec{K} = \eta\left\langle\vec{E}\right\rangle + \vec{L}\times\left\langle\vec{B}\right\rangle + \frac{\hat{n}}{2}\left(\left[\vec{E}\right]\cdot\left\langle\vec{D}\right\rangle - \left\langle\vec{E}\right\rangle\cdot\left[\vec{D}\right]\right)$$
$$+ \frac{\hat{n}}{2}\left(\left[\vec{H}\right]\cdot\left\langle\vec{B}\right\rangle - \left\langle\vec{H}\right\rangle\cdot\left[\vec{B}\right]\right) = \left[\overleftrightarrow{T}\right]\cdot\hat{n} + N\left[\vec{D}\times\vec{B}\right] \qquad (50a)$$

where $\left[\overleftrightarrow{T}\right]\cdot\hat{n}$ represents $\left[T_{ij}\right]n_j$, or

$$\vec{K} = \eta\left\langle\vec{E}\right\rangle + \vec{L}\times\left\langle\vec{B}\right\rangle + \frac{\hat{n}}{2}\left(\vec{E}^+\cdot\vec{D}^- - \vec{E}^-\cdot\vec{D}^+\right) + \frac{\hat{n}}{2}\left(\vec{H}^+\cdot\vec{B}^- - \vec{H}^-\cdot\vec{B}^+\right)$$
$$= \left[\vec{E}\left(\vec{D}\cdot\hat{n}\right) + \vec{H}\left(\vec{B}\cdot\hat{n}\right)\right] - \hat{n}\left[W\right] + N\left[\vec{D}\times\vec{B}\right] \qquad (50b)$$

where T_{ij} (eq. (45c)) has been substituted; this is the surface form of the electrodynamic momentum conservation law. The fourth component becomes

$$P = \vec{L} \cdot \left\langle \vec{E} \right\rangle + \frac{N}{2}\left(\left[\vec{E}\right] \cdot \left\langle \vec{D} \right\rangle - \left\langle \vec{E} \right\rangle \cdot \left[\vec{D}\right]\right) + \frac{N}{2}\left(\left[\vec{H}\right] \cdot \left\langle \vec{B} \right\rangle - \left\langle \vec{H} \right\rangle \cdot \left[\vec{B}\right]\right)$$

$$= -\hat{n} \cdot \left[\vec{E} \times \vec{H}\right] + N\left[W\right] \tag{51a}$$

or

$$P = \vec{L} \cdot \left\langle \vec{E} \right\rangle + \frac{N}{2}\left(\vec{E}^{+} \cdot \vec{D}^{-} - \vec{E}^{-} \cdot \vec{D}^{+}\right) + \frac{N}{2}\left(\vec{H}^{+} \cdot \vec{B}^{-} - \vec{H}^{-} \cdot \vec{B}^{+}\right)$$

$$= -\hat{n} \cdot \left[\vec{E} \times \vec{H}\right] + N\left[W\right] \tag{51b}$$

and this is the surface form of Poynting's theorem, or the conservation of electrodynamic energy. Tables III and IV display the volumetric and surface forms of the conservation equations in four and three dimensions.

Identities Derived From the Antisymmetric Part of the Stress-Energy Tensor

The stress-energy tensor $T_{\alpha\beta}$, as defined in set (45), is asymmetric; but it may be written as the sum of a symmetric and an antisymmetric tensor.

$$T_{\alpha\beta} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right) + \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right) \equiv T^{s}_{\alpha\beta} + T^{a}_{\alpha\beta} \tag{52a}$$

where, by equation (45b),

$$T^{a}_{\alpha\beta} = \begin{pmatrix} 0 & M_3 & -M_2 & -icR_1 \\ -M_3 & 0 & M_1 & -icR_2 \\ M_2 & -M_1 & 0 & -icR_3 \\ icR_1 & icR_2 & icR_3 & 0 \end{pmatrix} \tag{52b}$$

with

$$\vec{M} = \frac{1}{2}\left(\vec{E} \times \vec{D} + \vec{H} \times \vec{B}\right)$$

TABLE III.- MINKOWSKI CONSERVATION LAWS IN FOUR-DIMENSIONAL COVARIANT FORM

Volumetric (differential) form	Surface form at $\varphi(x,y,z,t) = 0$
$\kappa_\alpha = \frac{1}{c} F_{\alpha\theta}\Gamma_\theta + \frac{1}{4c}\left(F_{\theta\gamma}f_{\theta\gamma,\alpha} - F_{\theta\gamma,\alpha}f_{\theta\gamma}\right)$	$K_\alpha = \frac{1}{c}\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta + \frac{1}{4c}\frac{\varphi_{,\alpha}}{\lvert\nabla\varphi\rvert}\left(\left\langle F_{\theta\gamma}\right\rangle \left[f_{\theta\gamma}\right] - \left[F_{\theta\gamma}\right]\left\langle f_{\theta\gamma}\right\rangle\right)$
$= T_{\alpha\beta,\beta}$	$= \left[T_{\alpha\beta}\right]\frac{\varphi_{,\beta}}{\lvert\nabla\varphi\rvert}$

TABLE IV.- MINKOWSKI CONSERVATION LAWS IN THREE-DIMENSIONAL FORM

Volumetric (differential) form	Jump form at surface moving with speed N
$k_i = \rho E_i + \left(\vec{J} \times \vec{B}\right)_i + \frac{1}{2}\left(\vec{E}_{,i} \cdot \vec{D} - \vec{E} \cdot \vec{D}_{,i}\right) + \frac{1}{2}\left(\vec{H}_{,i} \cdot \vec{B} - \vec{H} \cdot \vec{B}_{,i}\right)$	$\vec{K} = \eta\left\langle\vec{E}\right\rangle + \vec{L} \times \left\langle\vec{B}\right\rangle + \frac{\hat{n}}{2}\left(\left[\vec{E}\right] \cdot \left\langle\vec{D}\right\rangle - \left\langle\vec{E}\right\rangle \cdot \left[\vec{D}\right]\right) + \frac{\hat{n}}{2}\left(\left[\vec{H}\right] \cdot \left\langle\vec{B}\right\rangle - \left\langle\vec{H}\right\rangle \cdot \left[\vec{B}\right]\right)$
$= T_{ij,j} - \left(\vec{D} \times \vec{B}\right)_{i,t}$	$= \left[\overleftrightarrow{T}\right] \cdot \hat{n} + N\left[\vec{D} \times \vec{B}\right]$
$p = \vec{J} \cdot \vec{E} + \frac{1}{2}\left(\vec{E} \cdot \vec{D}_{,t} - \vec{E}_{,t} \cdot \vec{D}\right) + \frac{1}{2}\left(\vec{H} \cdot \vec{B}_{,t} - \vec{H}_{,t} \cdot \vec{B}\right)$	$P = \vec{L} \cdot \left\langle\vec{E}\right\rangle + \frac{N}{2}\left(\left[\vec{E}\right] \cdot \left\langle\vec{D}\right\rangle - \left\langle\vec{E}\right\rangle \cdot \left[\vec{D}\right]\right) + \frac{N}{2}\left(\left[\vec{H}\right] \cdot \left\langle\vec{B}\right\rangle - \left\langle\vec{H}\right\rangle \cdot \left[\vec{B}\right]\right)$
$= -\mathrm{div}\left(\vec{E} \times \vec{H}\right) - W_{,t}$	$= -\hat{n} \cdot \left[\vec{E} \times \vec{H}\right] + N\left[W\right]$

$$\vec{R} = \frac{1}{2}\left(\vec{D} \times \vec{B} - \frac{1}{c^2}\,\vec{E} \times \vec{H}\right)$$

The antisymmetric part $T^a_{\alpha\beta}$ has been a subject of discussion since its inception by Minkowski in 1908. (See Møller (ref. 10, pp. 204-206) and Pauli (ref. 2, pp. 108-111).) Splitting the four-force densities also

$$\kappa_\alpha = T^s_{\alpha\beta,\beta} + T^a_{\alpha\beta,\beta} \equiv \kappa^s_\alpha + \kappa^a_\alpha \tag{53}$$

$$K_\alpha = \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[T^s_{\alpha\beta}\right] + \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[T^a_{\alpha\beta}\right] \equiv K^s_\alpha + K^a_\alpha \tag{54}$$

enables a number of relations between the antisymmetric parts $T^a_{\alpha\beta}$, κ^a_α, and K^a_α to be obtained by inspection of formally identical equations in tables I and II. These relations should provide a fresh vantage point for the symmetry issue.

Both $f_{\alpha\beta}$ (eq. (8c)) and $T^a_{\alpha\beta}$ (eq. (52b)) have the same antisymmetric form; utilizing equations (9a) and (53), (17) and (54), with definitions (8b) and (46), (10a) and (47a), the terms may be paired as shown:

Four-dimensional pairs:

$$\left.\begin{matrix} f_{\alpha\beta} & \Gamma_\alpha & \Lambda_\alpha \\ T^a_{\alpha\beta} & \kappa^a_\alpha & K^a_\alpha \end{matrix}\right\} \tag{55}$$

Three-dimensional pairs:

$$\left.\begin{matrix} \vec{H} & \vec{D} & \vec{J} & \rho & \vec{L} & \eta \\ \frac{1}{2}\left(\vec{E} \times \vec{D} + \vec{H} \times \vec{B}\right) & \frac{1}{2}\left(\vec{D} \times \vec{B} - \frac{1}{c^2}\,\vec{E} \times \vec{H}\right) & \vec{k}^a & \frac{1}{c^2}\,p^a & \vec{K}^a & \frac{1}{c^2}\,P^a \end{matrix}\right\} \tag{56}$$

Thus tables I and II go over by direct substitution to tables V and VI and give the tractions and power transfer (with related identities) contributed by the antisymmetrical part of the stress-energy tensor.

TABLE V.- ANTISYMMETRIC RELATIONS IN FOUR-DIMENSIONAL COVARIANT FORM

Volumetric (differential) form	Surface form at $\varphi(x,y,z,t) = 0$
$T^{a}_{\alpha\beta,\beta} = \kappa^{a}_{\alpha}$	$\left[T^{a}_{\alpha\beta}\right]\dfrac{\varphi_{,\beta}}{\lvert\nabla\varphi\rvert} = K^{a}_{\alpha};\ \varphi_{,\beta}K^{a}_{\beta} = 0$ (Corollary)
$\kappa^{a}_{\beta,\beta} = 0$	$\left[\kappa^{a}_{\beta}\right]\varphi_{,\beta} + \left(K^{a}_{\beta}\lvert\nabla\varphi\rvert\right)_{,\beta} = 0$

TABLE VI.- ANTISYMMETRIC RELATIONS IN THREE-DIMENSIONAL FORM

Volumetric (differential) form	Jump form at surface moving with speed N
$\operatorname{curl}\left(\vec{E}\times\vec{D} + \vec{H}\times\vec{B}\right) - \left(\vec{D}\times\vec{B} - \frac{1}{c^2}\vec{E}\times\vec{H}\right)_{,t} = 2\vec{K}^{a}$	$\hat{n}\times\left[\vec{E}\times\vec{D} + \vec{H}\times\vec{B}\right] + N\left[\vec{D}\times\vec{B} - \frac{1}{c^2}\vec{E}\times\vec{H}\right] = 2\vec{K}^{a}$
$\operatorname{div}\left(\vec{D}\times\vec{B} - \frac{1}{c^2}\vec{E}\times\vec{H}\right) = \frac{2}{c^2}p^{a}$	$\hat{n}\cdot\left[\vec{D}\times\vec{B} - \frac{1}{c^2}\vec{E}\times\vec{H}\right] = \frac{2}{c^2}p^{a}$
	(the two above) $\Bigg\}\ \hat{n}\cdot\vec{K}^{a} - \frac{1}{c^2}Np^{a} = 0$ (Corollary)
$\operatorname{div}\vec{k}^{a} + \frac{1}{c^2}p^{a}_{,t} = 0$	$\hat{n}\cdot\left[\vec{k}^{a}\right] - \frac{N}{c^2}\left[p^{a}\right] - \operatorname{div}_{surface}\left(\hat{n}\times\left(\hat{n}\times\vec{K}^{a}\right)\right) - \frac{1}{c^2}CNP^{a} + \frac{1}{c^2}\left(\frac{\partial}{\partial t} + N\frac{\partial}{\partial n}\right)P^{a} = 0$

APPENDIX A

TRANSFORMATION PROPERTIES OF THE SURFACE DENSITY QUASI-FOUR-VECTORS

The surface quasi-four-current density Λ_α and quasi-four-force density K_α are defined on a moving surface $\varphi(x,y,z,t) = 0$ by equations (10) and (47), respectively. Their behavior under the Lorentz transformation is apparent from equations (17) and (48), since $\left[f_{\alpha\beta}\right]\varphi_{,\beta}$ and $\left[T_{\alpha\beta}\right]\varphi_{,\beta}$ are clearly four-vectors and, consequently, $\Lambda_\alpha|\nabla\varphi|$ and $K_\alpha|\nabla\varphi|$ must be also. Thus, if x,y,z,t and x',y',z',t' are two inertial frames related by the Lorentz transformation

$$\left.\begin{aligned} \Lambda_\alpha'|\nabla'\varphi'| &= l_{\alpha\beta}\Lambda_\beta|\nabla\varphi| \\ \Lambda_\alpha|\nabla\varphi| &= \Lambda_\beta'|\nabla'\varphi'|\,l_{\beta\alpha} \end{aligned}\right\} \tag{A1}$$

with identical relations for K_α, where

$$\varphi(x,y,z,t) = \varphi'(x',y',z',t')$$

$$\nabla' = \hat{i}_1 \frac{\partial}{\partial x'} + \hat{i}_2 \frac{\partial}{\partial y'} + \hat{i}_3 \frac{\partial}{\partial z'}$$

and $l_{\alpha\beta}$ is the Lorentz transformation matrix. (See Møller (ref. 10, pp. 94 and 118).) These are the surface density transformation relations.

It also follows from the scalar invariance of $|\Box\varphi|$ and from equation (5b), that $\Lambda_\alpha \dfrac{|\nabla\varphi|}{|\Box\varphi|} = \dfrac{\Lambda_\alpha}{\sqrt{1 - N^2/c^2}}$ and $K_\alpha \dfrac{|\nabla\varphi|}{|\Box\varphi|} = \dfrac{K_\alpha}{\sqrt{1 - N^2/c^2}}$ are also four-vectors. Hence the transformation relations may alternately be written

$$\left.\begin{aligned} \frac{\Lambda_\alpha'}{\sqrt{1 - N'^2/c^2}} &= l_{\alpha\beta}\frac{\Lambda_\beta}{\sqrt{1 - N^2/c^2}} \\ \frac{\Lambda_\alpha}{\sqrt{1 - N^2/c^2}} &= \frac{\Lambda_\beta'}{\sqrt{1 - N'^2/c^2}}\,l_{\beta\alpha} \end{aligned}\right\} \tag{A2}$$

APPENDIX B

APPLICATION OF STOKES' THEOREM TO THE CURRENT BOUNDARY CONDITION

For a closed two-dimensional hypercircuit on a hypersurface (which is taken here to be $\varphi(x,y,z,t) = 0$), Stokes' theorem states (Synge and Schild (ref. 11, p. 269))

$$\oint\!\!\oint ds_{\mu\nu}\psi_{\mu\nu} = \iiint \psi_{\alpha\beta,\gamma}\, d\sigma_{\alpha\beta\gamma} \tag{B1}$$

where $\psi_{\mu\nu}$ is an arbitrary set of functions of the coordinates (a tensor for the present purposes), and, in terms of the orthogonal set of line elements (29),

$$d\sigma_{\alpha\beta\gamma} = \begin{vmatrix} \mu_\alpha\mu_\epsilon\, dx_\epsilon & \delta x_\alpha & \Delta x_\alpha \\ \mu_\beta\mu_\epsilon\, dx_\beta & \delta x_\beta & \Delta x_\beta \\ \mu_\gamma\mu_\epsilon\, dx_\epsilon & \delta x_\gamma & \Delta x_\gamma \end{vmatrix} \qquad ds_{\mu\nu} = \begin{vmatrix} \delta x_\mu & \Delta x_\mu \\ \delta x_\nu & \Delta x_\nu \end{vmatrix} \tag{B2}$$

By introducing the dual four-vectors (Møller (ref. 10, pp. 112-117))

$$\left.\begin{aligned} \Psi_\theta^* &= \frac{1}{3!}\,\delta_{\theta\lambda\mu\nu}\psi_{\lambda\mu,\nu} \\ d\sigma_\theta^* &= \frac{1}{3!}\,\delta_{\theta\alpha\beta\gamma}\, d\sigma_{\alpha\beta\gamma} \end{aligned}\right\} \tag{B3}$$

the right-hand side of theorem (B1) may be rewritten

$$\oint\!\!\oint ds_{\mu\nu}\psi_{\mu\nu} = 3!\iiint \Psi_\theta^* d\sigma_\theta^* \tag{B4}$$

APPENDIX B

The element $d\sigma_\theta^*$ may be further reduced by equations (B2), (31), and (30a) to yield

$$d\sigma_\theta^* = \delta_{\theta\alpha\beta\gamma}\left(\mu_\alpha\mu_\epsilon\ dx_\epsilon\right)\delta x_\beta \Delta x_\gamma = m_\theta\ d\sigma \tag{B5}$$

where m_θ is the unit four-normal on $\varphi = 0$, and equation (B4) becomes

$$\oiint ds_{\mu\nu}\psi_{\mu\nu} = \iiint d\sigma\ m_\theta \delta_{\theta\lambda\mu\nu}\psi_{\mu\nu,\lambda}$$

which is the form of Stokes' theorem given in equation (35).

Using theorem (35) in equation (34), with $\psi_{\mu\nu} = \delta_{\mu\nu\alpha\beta}\Lambda_\alpha m_\beta \dfrac{|\nabla\varphi|}{|\Box\varphi|}$ gives

$$\iiint \left[\Gamma_\alpha\right] m_\alpha\ d\sigma - \frac{1}{2}\iiint d\sigma\ m_\theta\delta_{\theta\lambda\mu\nu}\delta_{\mu\nu\alpha\beta}\left(\Lambda_\alpha m_\beta \frac{|\nabla\varphi|}{|\Box\varphi|}\right)_{,\lambda} = 0 \tag{B6}$$

But

$$\delta_{\theta\lambda\mu\nu}\delta_{\mu\nu\alpha\beta} = 2\left(\delta_{\theta\alpha}\delta_{\lambda\beta} - \delta_{\lambda\alpha}\delta_{\theta\beta}\right) \tag{B7}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Hence

$$\iiint \left[\Gamma_\alpha\right] m_\alpha\ d\sigma - \iiint d\sigma \left\{ m_\alpha\left(\Lambda_\alpha m_\beta \frac{|\nabla\varphi|}{|\Box\varphi|}\right)_{,\beta} - m_\beta\left(\Lambda_\alpha m_\beta \frac{|\nabla\varphi|}{|\Box\varphi|}\right)_{,\alpha} \right\} = 0 \tag{B8}$$

Since $m_\alpha\Lambda_\alpha = \varphi_{,\alpha}\Lambda_\alpha = 0$ by corollary (18) and $m_\beta m_\beta = 1$ by equation (4)

$$\iiint \left[\Gamma_\alpha\right] m_\alpha\ d\sigma - \iiint d\sigma \left\{ \frac{|\nabla\varphi|}{|\Box\varphi|}\Lambda_\alpha m_\beta\left(m_{\beta,\alpha} - m_{\alpha,\beta}\right) - \left(\Lambda_\alpha \frac{|\nabla\varphi|}{|\Box\varphi|}\right)_{,\alpha} \right\} = 0 \tag{B9}$$

But

$$m_{\beta,\alpha} - m_{\alpha,\beta} = \left(\frac{\varphi_{,\beta}}{|\Box\varphi|}\right)_{,\alpha} - \left(\frac{\varphi_{,\alpha}}{|\Box\varphi|}\right)_{,\beta} = \frac{1}{|\Box\varphi|^2}\left(\varphi_{,\alpha}|\Box\varphi|_{,\beta} - \varphi_{,\beta}|\Box\varphi|_{,\alpha}\right) \tag{B10}$$

Therefore

$$\iiint \left[\Gamma_\alpha\right] m_\alpha \, d\sigma + \iiint d\sigma \left\{\frac{|\nabla\varphi|}{|\Box\varphi|^2}\Lambda_\alpha|\Box\varphi|_{,\alpha} + \left(\Lambda_\alpha \frac{|\nabla\varphi|}{|\Box\varphi|}\right)_{,\alpha}\right\} = 0 \tag{B11}$$

or

$$\iiint \left[\Gamma_\alpha\right] m_\alpha \, d\sigma + \iiint d\sigma \frac{\left(\Lambda_\alpha|\nabla\varphi|\right)_{,\alpha}}{|\Box\varphi|} = 0 \tag{B12}$$

and result (36) is thus established.

APPENDIX C

VERIFICATION OF IDENTITY (39b)

Identity (39b) is most easily checked by manipulating the right-hand side as shown below:

$$\frac{1}{|\nabla\varphi|}\left(\Lambda_\alpha|\nabla\varphi|\right)_{,\alpha} = I_1 + I_2$$

where

$$I_1 = \left(\frac{\partial}{\partial t} + N\frac{\partial}{\partial n}\right)\eta + N\eta \operatorname{div}\hat{n}$$

$$I_2 = \hat{n}\cdot\operatorname{curl}\left(\hat{n}\times\vec{L}\right)$$

Combining terms in I_1

$$I_1 = \hat{n}\cdot\frac{\partial}{\partial t}\left(\hat{n}\eta\right) + N \operatorname{div}\left(\hat{n}\eta\right) \tag{C1}$$

and substituting equations (1) and (2) gives,

$$I_1 = \frac{\varphi_{,j}}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\eta\right)_{,t} - \frac{\varphi_{,t}}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\eta\right)_{,j} \tag{C2}$$

Because $x_4 = ict$ (eq. (8a)) and $\Lambda_4 = ic\eta$ (eq. (10a))

$$I_1 = \frac{\varphi_{,j}}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\Lambda_4\right)_{,4} - \frac{\varphi_{,4}}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\Lambda_4\right)_{,j}$$

$$= \frac{1}{|\nabla\varphi|}\left(|\nabla\varphi|\Lambda_4\right)_{,4} - \frac{1}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\varphi_{,4}\Lambda_4\right)_{,j} \tag{C3}$$

and by corollary (18)

$$I_1 = \frac{1}{|\nabla\varphi|}\left(|\nabla\varphi|\Lambda_4\right)_{,4} + \frac{1}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\varphi_{,i}\Lambda_i\right)_{,j} \tag{C4}$$

In tensor notation, I_2 is given by

$$I_2 = \frac{\varphi_{,i}}{|\nabla\varphi|}\epsilon_{ijk}\left(\epsilon_{klm}\frac{\varphi_{,l}}{|\nabla\varphi|}\Lambda_m\right)_{,j} \tag{C5}$$

where the Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} +1(-1) & \text{(for } ijk \text{ an even (odd) permutation of 123)} \\ 0 & \text{(for two or more indices alike)} \end{cases} \tag{C6}$$

It follows that

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{C7}$$

where δ_{ij} is the Kronecker delta, and equation (C5) becomes

$$\begin{aligned} I_2 &= \frac{\varphi_{,i}}{|\nabla\varphi|}\left(\frac{\varphi_{,i}}{|\nabla\varphi|}\Lambda_j\right)_{,j} - \frac{\varphi_{,i}}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\Lambda_i\right)_{,j} \\ &= \frac{1}{|\nabla\varphi|}\left(|\nabla\varphi|\Lambda_j\right)_{,j} - \frac{1}{|\nabla\varphi|}\left(\frac{\varphi_{,j}}{|\nabla\varphi|}\varphi_{,i}\Lambda_i\right)_{,j} \end{aligned} \tag{C8}$$

When equations (C4) and (C8) are added, the second terms of each cancel

$$\begin{aligned} I_1 + I_2 &= \frac{1}{|\nabla\varphi|}\left(|\nabla\varphi|\Lambda_j\right)_{,j} + \frac{1}{|\nabla\varphi|}\left(|\nabla\varphi|\Lambda_4\right)_{,4} \\ &= \frac{1}{|\nabla\varphi|}\left(|\nabla\varphi|\Lambda_\alpha\right)_{,\alpha} \end{aligned} \tag{C9}$$

and identity (39b) is thus established.

APPENDIX D

ESTABLISHMENT OF THE CENTER EXPRESSION OF EQUATION (49a) BY MEANS OF THE BOUNDARY CONDITIONS

The following two identities are easily proved

$$\left[Ff\right] = \left[F\right]\left\langle f\right\rangle + \left\langle F\right\rangle\left[f\right] \tag{D1}$$

$$F^{+}f^{-} - F^{-}f^{+} = \left[F\right]\left\langle f\right\rangle - \left\langle F\right\rangle\left[f\right] \tag{D2}$$

where the notation is defined by equations (49c). The center expression of equation (49a) may be validated most easily by deriving the surface conservation laws from the boundary conditions by the procedure illustrated in Møller (ref. 10, p. 202) for obtaining the volumetric conservation laws from Maxwell's equations.

Starting from the identity

$$\left\langle F_{\alpha\theta}\right\rangle \Lambda_{\theta} = \left\langle F_{\alpha\theta}\right\rangle \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[f_{\theta\beta}\right] \tag{D3}$$

based on boundary condition (17) and substituting equation (D1) gives

$$\left\langle F_{\alpha\theta}\right\rangle \Lambda_{\theta} = \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[F_{\alpha\theta}f_{\theta\beta}\right] - \frac{\varphi_{,\beta}}{2|\nabla\varphi|}\left[F_{\alpha\theta}\right]\left\langle f_{\theta\beta}\right\rangle - \frac{\varphi_{,\beta}}{2|\nabla\varphi|}\left[F_{\alpha\theta}\right]\left\langle f_{\theta\beta}\right\rangle \tag{D4}$$

The last term may be manipulated as shown

$$-\frac{\varphi_{,\beta}}{2|\nabla\varphi|}\left[F_{\alpha\theta}\right]\left\langle f_{\theta\beta}\right\rangle = -\frac{\varphi_{,\beta}}{2|\nabla\varphi|}\left[F_{\theta\alpha}\right]\left\langle f_{\beta\theta}\right\rangle = -\frac{\varphi_{,\theta}}{2|\nabla\varphi|}\left[F_{\beta\alpha}\right]\left\langle f_{\theta\beta}\right\rangle$$

so that equation (D4) becomes

$$\left\langle F_{\alpha\theta}\right\rangle \Lambda_{\theta} = -\frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[F_{\alpha\theta}f_{\beta\theta}\right] - \frac{1}{2}\left(\frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[F_{\alpha\theta}\right] + \frac{\varphi_{,\theta}}{|\nabla\varphi|}\left[F_{\beta\alpha}\right]\right)\left\langle f_{\theta\beta}\right\rangle \tag{D5}$$

Upon substitution of boundary condition (25)

$$\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta = -\frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[F_{\alpha\theta}f_{\beta\theta}\right] + \frac{\varphi_{,\alpha}}{4|\nabla\varphi|}\left[F_{\theta\beta}\right]\left\langle f_{\theta\beta}\right\rangle + \frac{\varphi_{,\alpha}}{4|\nabla\varphi|}\left[F_{\theta\beta}\right]\left\langle f_{\theta\beta}\right\rangle \tag{D6}$$

and by identity (D1)

$$\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta = -\frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[F_{\alpha\theta}f_{\beta\theta}\right] + \frac{\varphi_{,\alpha}}{4|\nabla\varphi|}\left[F_{\theta\gamma}f_{\theta\gamma}\right] + \frac{\varphi_{,\alpha}}{4|\nabla\varphi|}\left(\left[F_{\theta\beta}\right]\left\langle f_{\theta\beta}\right\rangle - \left\langle F_{\theta\beta}\right\rangle\left[f_{\theta\beta}\right]\right) \tag{D7}$$

Hence, upon division by c

$$\frac{1}{c}\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta = -\frac{\varphi_{,\beta}}{c|\nabla\varphi|}\left[F_{\alpha\theta}f_{\beta\theta}\right] + \frac{\delta_{\alpha\beta}}{4c}\frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[F_{\theta\gamma}f_{\theta\gamma}\right] + \frac{1}{4c}\frac{\varphi_{,\alpha}}{|\nabla\varphi|}\left(\left[F_{\theta\beta}\right]\left\langle f_{\theta\beta}\right\rangle - \left\langle F_{\theta\beta}\right\rangle\left[f_{\theta\beta}\right]\right) \tag{D8}$$

and with terms transposed

$$\frac{1}{c}\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta + \frac{1}{4c}\frac{\varphi_{,\alpha}}{|\nabla\varphi|}\left(\left\langle F_{\theta\beta}\right\rangle\left[f_{\theta\beta}\right] - \left[F_{\theta\beta}\right]\left\langle f_{\theta\beta}\right\rangle\right) = \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[-\frac{1}{c}F_{\alpha\theta}f_{\beta\theta} + \frac{\delta_{\alpha\beta}}{4c}F_{\theta\gamma}f_{\theta\gamma}\right] \tag{D9}$$

or by definition (45a)

$$\frac{1}{c}\left\langle F_{\alpha\theta}\right\rangle \Lambda_\theta + \frac{1}{4c}\frac{\varphi_{,\alpha}}{|\nabla\varphi|}\left(\left\langle F_{\theta\beta}\right\rangle\left[f_{\theta\beta}\right] - \left[F_{\theta\beta}\right]\left\langle f_{\theta\beta}\right\rangle\right) = \frac{\varphi_{,\beta}}{|\nabla\varphi|}\left[T_{\alpha\beta}\right] \tag{D10}$$

which is the identity in question.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., September 18, 1964.

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